Structure of infinitely divisible semimartingales

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Abstract

This paper gives a complete characterization of infinitely divisible semimartingales, i.e., semimartingales whose finite dimensional distributions are infinitely divisible. An explicit and essentially unique decomposition of such semimartingales is obtained. A new approach, combining series decompositions of infinitely divisible processes with detailed analysis of their jumps, is presented. As an ilustration of the main result, the semimartingale property is explicitely determined for a large class of stationary increment processes and several examples of processes of interest are considered. These results extend Stricker's theorem characterizing Gaussian semimartingales and Knight's theorem describing Gaussian moving average semimartingales, in particular.

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1 Introduction

Semimartingales play a crucial role in stochastic analysis as they form the class of *good integrators* for the Itô type stochastic integral, cf. the Bichteler–Dellacherie Theorem [10], see also Beiglböck et al. [8] for a direct proof. Semimartingales also play a fundamental role in mathematical finance. Roughly

speaking, the (discounted) asset price process must be a semimartingale in order to preclude arbitrage opportunities, see Beiglböck et al. [8, Theorems 1.4, 1.6] for details, see also [20]. The question whether a given process is a semimartingale is also of importance in stochastic modeling, where long memory processes with possible jumps and high volatility are considered as driving processes for stochastic differential equations. Examples of such processes include various fractional, or more generally, Volterra processes driven by Lévy processes.

The problem of identifying semimartingales within given classes of stochastic processes has a long history. In the context of Gaussian processes, it was intensively studied in 1980s. Gal'chuk [16] investigated Gaussian semimartingales addressing a question posed by Prof. A.N. Shiryayev. Key results on Gaussian semimartingales are due to Jain and Monrad [18, Theorem 1.8] and Stricker [28, Théorème 1], see also Liptser and Shiryayev [22, Ch. 4.9] and [15, 2, 4, 16]. A particularly interesting result is due to Knight [21, Theorem 6.5]: Let $\mathbf{X} = (X_t)_{t\geq 0}$ be a Gaussian moving average of the form

$$X_t = \int_{-\infty}^{t} \phi(t-s)\, dW_s,$$

where $(W_t)_{t\in\mathbb{R}}$ is a Brownian motion and $\phi\colon \mathbb{R} \to \mathbb{R}$ is a Lebesgue square integrable deterministic function vanishing on the negative axis. Then $\mathbf{X}$ is a semimartingale with respect to the filtration $(\mathcal{F}_t^W)_{t\geq 0}$ if and only if ϕ is absolutely continuous on $\mathbb{R}_+$ with a square integrable derivative. Then, $\mathbf{X}$ can be decomposed uniquely into a Brownian motion and a predictable process of finite variation. Extensions of Knight's result were given by Jeulin and Yor [19], see also [13, 12, 1, 5].

The class of infinitely divisible processes is a natural extension of the class of Gaussian processes. A process $\mathbf{X} = (X_t)_{t\geq 0}$ is said to be infinitely divisible if its all finite dimensional distributions are infinitely divisible. This work is aimed to determine the structure of infinitely divisible semimartingales.

Recall that a semimartingale $\mathbf{X} = (X_t)_{t\geq 0}$, by definition, admits a decomposition

$$X_t = X_0 + M_t + A_t, \quad t \geq 0, \tag{1.1}$$

where $\mathbf{M} = (M_t)_{t\geq 0}$ is a local martingale and $\mathbf{A} = (A_t)_{t\geq 0}$ is a process of finite variation. The problem of identifying infinitely divisible semimartingales can be divided into two questions for an infinitely divisible process $\mathbf{X} = (X_t)_{t\geq 0}$:

Q1. Assuming that $\mathbf{X}$ is a semimartingale, what is the structure of the

processes $\mathbf{M}$ and $\mathbf{A}$ in its decomposition (1.1)? In particular, do they have to be infinitely divisible?

Q2. How to verify whether or not a given infinitely divisible process $\mathbf{X}$ is a semimartingale?

The family of infinitely divisible processes constitutes a huge class, so we will focus on a parametrization that will be convenient to state our results as well as for applications. We will assume that a primary description of an infinitely divisible processes $\mathbf{X} = (X_t)_{t\geq 0}$ is its 'spectral' representation of the form

$$X_t = \int_{(-\infty,t]\times V} \phi(t,s,v)\,\Lambda(ds,dv),$$

where V is a set, Λ is an independently scattered infinitely divisible random measure on $\mathbb{R}\times V$, and $\phi\colon \mathbb{R}^2\times V \to \mathbb{R}$ is a measurable deterministic function (see Section 2 for details). If we further assume that $\phi(t,s,x) = 0$ whenever $s > t$, then $\mathbf{X}$ is adapted to the filtration $\mathbb{F}^\Lambda = (\mathcal{F}_t^\Lambda)_{t\geq 0}$, where $\mathcal{F}_t^\Lambda$ is the σ-algebra generated by Λ restricted to $(-\infty,t]\times V$. We characterize the semimartingale property of $\mathbf{X}$ with respect to $\mathbb{F}^\Lambda$, which, in the spirit, is similar to the above mentioned results of Stricker and Knight. However, in the non Gaussian case, our methods are completely different. We combine series representations of càdlàg infinitely divisible processes with detailed analysis of their jumps, which is a novel approach as far as we know. This is possible because such series representations converge uniformly a.s. on compacts, as shown in Basse-O'Connor and Rosiński [7, Theorem 3.1].

Section 2 contains preliminary definitions and facts. Our main result, Theorem 3.1, is stated and proved in Section 3. It gives the necessary and sufficient conditions for $(\mathbf{X},\mathbb{F}^\Lambda)$ to be a semimartingale, together with an essentially unique explicit decomposition of $\mathbf{X}$ into infinitely divisible components. It completely answers the above question Q1 and gives a framework how to approach question Q2 in concrete situations. In Section 4 we obtain explicit necessary and sufficient conditions for a large class of stationary increment infinitely divisible processes to be semimartingales. These conditions generalize, in a natural way, findings in Basse and Pedersen [3]. We conclude this paper with examples showing how these conditions can be verified for several processes of interest.

2 Preliminaries

Throughout the paper $(\Omega, \mathcal{F}, \mathbb{P})$ stands for a probability space and $(V, \mathcal{V})$ denotes a countably generated measurable space. Consider a process $\mathbf{X} = (X_t)_{t\geq 0}$ of the form

$$X_t = \int_{(-\infty,t]\times V} \phi(t,s,v)\,\Lambda(ds,dv) \tag{2.1}$$

where $\phi\colon \mathbb{R}^2 \times V \to \mathbb{R}$ is a measurable deterministic function such that for every $(s,v) \in \mathbb{R}\times V$, $\phi(\cdot,s,v)$ is càdlàg and Λ is an independently scattered infinitely divisible random measure on a σ-ring $\mathscr{S}$ of subsets of $\mathbb{R}\times V$ such that

$$\big\{[a,b]\times B:\ [a,b]\subset\mathbb{R},\ B\in\mathcal{V}_0\big\}\subset\mathscr{S}\subset\mathscr{B}(\mathbb{R})\otimes\mathcal{V},$$

for some countable ring $\mathcal{V}_0$ generating $\mathcal{V}$. It follows that $\sigma(\mathscr{S}) = \mathscr{B}(\mathbb{R})\otimes\mathcal{V}$. For example, Λ can be a Poisson random measure. In general, we assume that for every $A\in\mathscr{S}$, $\Lambda(A)$ has an infinitely divisible distribution with the cumulant

$$\log \mathbb{E}e^{i\theta\Lambda(A)} = \int_A \Big[i\theta b(u) - \frac{1}{2}\theta^2\sigma^2(u) + \int_{\mathbb{R}} \big(e^{i\theta x} - 1 - i\theta[\![x]\!]\big)\,\rho_u(dx)\Big]\,\kappa(du), \tag{2.2}$$

where $u = (s,v)\in\mathbb{R}\times V$, $b\colon \mathbb{R}\times V\to\mathbb{R}$ is a measurable function, κ is a σ-finite measure on $\mathbb{R}\times V$, $\{\rho_u\}_{u\in\mathbb{R}\times V}$ is a measurable family of Lévy measures on $\mathbb{R}$, and

$$[\![x]\!] = \frac{x}{|x|\vee 1} = \begin{cases} x & \text{if } |x|<1\,, \\ \operatorname{sgn}(x) & \text{otherwise} \end{cases} \tag{2.3}$$

is a truncation function. For example, if Λ is a Poisson random measure, then $b(u) = 1$, $\sigma^2(u) = 0$, and $\rho_u = \delta_1$. The integral in (2.1) is defined as in Rajput and Rosiński [25]. Accordingly to [25, Theorem 2.7], given a measurable deterministic function f, the integral $\int_{\mathbb{R}\times V} f(u)\,\Lambda(du)$ exists if and only if

(a) $\int_{\mathbb{R}\times V} |B(f(u),u)|\,\kappa(du) < \infty$,

(b) $\int_{\mathbb{R}\times V} K(f(u),u)\,\kappa(du) < \infty$,

where

$$B(x,u) = xb(u) + \int_{\mathbb{R}} \left([\![xy]\!] - x[\![y]\!] \right) \rho_u(dy) \quad \text{and} \tag{2.4}$$

$$K(x,u) = x^2\sigma^2(u) + \int_{\mathbb{R}} [\![xy]\!]^2 \, \rho_u(dy), \qquad x \in \mathbb{R},\ u \in \mathbb{R} \times V.$$

The process $\mathbf{X}$ in (2.1) is infinitely divisible, i.e., all its finite dimensional distributions are infinitely divisible.

We will further require that

$$\Lambda(\{s\} \times B) = 0 \quad \text{a.s. for every } s \in \mathbb{R},\ B \in \mathcal{V}_0. \tag{2.5}$$

This condition is equivalent to that $\kappa(\{s\} \times V) = 0$ for every $s \in \mathbb{R}$.

Let $\mathbb{F}^\Lambda = (\mathcal{F}_t^\Lambda)_{t \geq 0}$ be the augmented filtration generated by Λ, i.e. $\mathbb{F}^\Lambda$ is the least filtration satisfying the *usual conditions* of right-continuity and completeness such that

$$\sigma\Big(\Lambda(A) : A \in \mathscr{S},\ A \subseteq (-\infty, t] \times V\Big) \subseteq \mathcal{F}_t^\Lambda, \qquad t \geq 0.$$

Definition (semimartingale): Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be a filtration. An $\mathbb{F}$-adapted process $\mathbf{X} = (X_t)_{t \geq 0}$ is a semimartingale with respect to $\mathbb{F}$ if it admits a decomposition

$$X_t = X_0 + A_t + M_t, \qquad t \geq 0, \tag{2.6}$$

where $\mathbf{M} = (M_t)_{t \geq 0}$ is a càdlàg local martingale with respect to $\mathbb{F}$, $\mathbf{A} = (A_t)_{t \geq 0}$ is an $\mathbb{F}$-adapted process with càdlàg paths of finite variation and $A_0 = M_0 = 0$. (Càdlàg means right-continuous with left-hand limits.) Moreover, $\mathbf{X}$ is called a *special semimartingale* if, in addition, $\mathbf{A}$ in (2.6) can be chosen predictable. In the later case, the decomposition (2.6) is unique and is called the *canonical decomposition* of $\mathbf{X}$ and processes $\mathbf{M}$ and $\mathbf{A}$ are called the martingale and finite variation components, respectively. We refer to Jacod and Shiryaev [17] and Protter [24] for basic properties of semimartingales.

For stochastic processes $\mathbf{X} = (X_t)_{t \geq 0}$ and $\mathbf{Y} = (Y_t)_{t \geq 0}$ we will write $\mathbf{X} = \mathbf{Y}$ when $\mathbf{X}$ and $\mathbf{Y}$ are indistinguishable, i.e., $\{\omega : X_t(\omega) \neq Y_t(\omega) \text{ for some } t \geq 0\}$ is a $\mathbb{P}$-null set. For each càdlàg function $g \colon \mathbb{R}_+ \to \mathbb{R}$ let $\Delta g(t) = \lim_{s \uparrow t, s < t}(g(t) - g(s))$ denote the jump of g at $t > 0$ and $\Delta g(0) = 0$.

3 Infinitely divisible semimartingales

The following is our main result.

Theorem 3.1. *Let* $\mathbf{X} = (X_t)_{t\geq 0}$ *be a process of the form*

$$X_t = \int_{(-\infty,t]\times V} \phi(t,s,v)\,\Lambda(ds,dv),$$

specified by (2.1)–(2.3), *and let* B *be given by* (2.4). *Assume that for every* $t>0$

$$\int_{(0,t]\times V} \big|B\big(\phi(s,s,v),(s,v)\big)\big|\,\kappa(ds,dv) < \infty. \tag{3.1}$$

Then $\mathbf{X}$ *is a semimartingale with respect to the filtration* $\mathbb{F}^\Lambda = (\mathcal{F}_t^\Lambda)_{t\geq 0}$ *if and only if*

$$X_t = X_0 + M_t + A_t, \quad t \geq 0, \tag{3.2}$$

where $\mathbf{M} = (M_t)_{t\geq 0}$ *is a continuous in probability semimartingale with independent increments given by*

$$M_t = \int_{(0,t]\times V} \phi(s,s,v)\,\Lambda(ds,dv), \quad t\geq 0 \tag{3.3}$$

(i.e., the integral exists), and $\mathbf{A} = (A_t)_{t\geq 0}$ *is a predictable càdlàg process of finite variation given by*

$$A_t = \int_{(-\infty,t]\times V} [\phi(t,s,v) - \phi(s_+,s,v)]\,\Lambda(ds,dv). \tag{3.4}$$

Decomposition (3.2) *is unique in the following sense: If* $\mathbf{X} = X_0 + \mathbf{M}' + \boldsymbol{A}'$, *where* $\mathbf{M}'$ *is a continuous in probability semimartingale with independent increments and* $\boldsymbol{A}'$ *is a predictable càdlàg process of finite variation, then* $\mathbf{M}' = \mathbf{M} + g$ *and* $\boldsymbol{A}' = \boldsymbol{A} - g$ *for some continuous deterministic function* g *of finite variation, with* $\mathbf{M}$ *and* $\boldsymbol{A}$ *given by* (3.3) *and* (3.4).

If $\mathbf{X}$ *is a semimartingale, then it is a special semimartingale if and only if* $\mathbb{E}|M_t| < \infty$ *for all* $t>0$*; and in this case* $(M_t - \mathbb{E}M_t)_{t\geq 0}$ *is a martingale and*

$$X_t = X_0 + (M_t - \mathbb{E}M_t) + (A_t + \mathbb{E}M_t), \quad t\geq 0$$

is the canonical decomposition of $\mathbf{X}$.

Remark 3.2. Condition (3.1) is always satisfied when Λ is symmetric. Indeed, in this case $B = 0$.

Remark 3.3. Stricker's theorem [28] says that Gaussian semimartingales are those who admit a decomposition into a Gaussian martingale with independent increments and a predictable process of finite variation. Our theorem is an exact extension of this result to the infinitely divisible case.

Remark 3.4. There is a slight inconsistency in the notations used in (1.1) and (3.2). In (3.2) $\mathbf{M}$ is a semimartingale with independent increments, which is a martingale when centered, in which case (1.1) and (3.2) coincide. However, if $\mathbf{M}$ does not have zero expectation, then further decomposition of $\mathbf{M}$ into a martingale and a process of finite variation is needed to obtain (1.1), but this is standard, see e.g. [27, Theorem 19.2].

Example 3.5. Consider the setting in Theorem 3.1 and suppose that Λ is an α-stable random measure and $\alpha \in (0, 1)$. Then $\mathbf{X}$ is a semimartingale with respect to $\mathbb{F}^\Lambda$ if and only if it is of finite variation. This follows by Theorem 3.1 because the process $\mathbf{M}$ given by (4.6) is of finite variation.

Indeed, the Lévy–Itô decomposition, see [27, Theorem 19.3], yields that $\mathbf{M} = \bar{\mathbf{M}} + \mathbf{a}$, where $\mathbf{a}$ is a continuous deterministic function and process $\bar{\mathbf{M}}$ is of finite variation, see [27, Eq. (20.24)]. Moreover since $\mathbf{M}$ is a semimartingale, the function $\mathbf{a}$ is of finite variation, cf. [17, Ch. II, Corollary 5.11], which implies that $\mathbf{M}$ is of finite variation.

Proof of Theorem 3.1. The sufficiency is obvious. To show the necessary part we need to show that a semimartingale $\mathbf{X}$ has a decomposition (3.2) where the processes $\mathbf{M}$ and $\mathbf{A}$ have the stated properties. We will start by considering the case where Λ does not have a Gaussian component, i.e. $\sigma^2 = 0$.

Case 1. Λ has no Gaussian component: We divide the proof into the following six steps.

Step 1: Let $X_t^0 = X_t - \beta(t)$, with

$$\beta(t) = \int_U B\big(\phi(t,u), u\big)\, \kappa(du).$$

We will give the series representation for $\mathbf{X}^0$ that will be crucial for our considerations. To this end, define for $s \neq 0$ and $u \in U = \mathbb{R} \times V$

$$R(s,u) = \begin{cases} \inf\{x > 0 : \rho_u(x,\infty) \leq s\} & \text{if } s > 0, \\ \sup\{x < 0 : \rho_u(-\infty,x) \leq -s\} & \text{if } s < 0. \end{cases}$$

Choose a probability measure $\tilde{\kappa}$ on U equivalent to κ, and let $h(u) = \frac{1}{2}(d\tilde{\kappa}/d\kappa)(u)$. By an extension of our probability space if necessary, Rosiński [26], Proposition 2 and Theorem 4.1, shows that there exists three independent sequences $(\Gamma_i)_{i\in\mathbb{N}}$, $(\epsilon_i)_{i\in\mathbb{N}}$, and $(T_i)_{i\in\mathbb{N}}$, where Γ_i are partial sums of i.i.d. standard exponential random variables, ϵ_i are i.i.d. symmetric Bernoulli random variables, and $T_i = (T_i^1, T_i^2)$ are i.i.d. random variables in U with the common distribution $\tilde{\kappa}$, such that for every $A \in \mathscr{S}$,

$$\Lambda(A) = \nu_0(A) + \sum_{j=1}^{\infty}\Big[R_j \mathbf{1}_A(T_j) - \nu_j(A)\Big] \qquad \text{a.s.} \tag{3.5}$$

where $R_j = R(\epsilon_j \Gamma_j h(T_j), T_j)$, $\nu_0(A) = \int_A b(u)\,\kappa(du)$, and for $j \geq 1$

$$\nu_j(A) = \int_{\Gamma_{j-1}}^{\Gamma_j} \mathbb{E}[\![R(\epsilon_1 r h(T_1), T_1)]\!]\mathbf{1}_A(T_1)\,dr.$$

It follows by the same argument that

$$X_t^0 = \sum_{j=1}^{\infty}\big[R_j \phi(t,T_j) - \alpha_j(t)\big] \qquad \text{a.s.,}$$

where

$$\alpha_j(t) = \int_{\Gamma_{j-1}}^{\Gamma_j} \mathbb{E}[\![R(\epsilon_1 r h(T_1), T_1)\phi(t,T_1)]\!]\,dr.$$

Step 2: We will show that for every $i \in \mathbb{N}$

$$\Delta X_{T_i^1} = R_i \phi(T_i^1, T_i) \quad \text{a.s.} \tag{3.6}$$

Since $\mathbf{X}$ is càdlàg, the series

$$X_t^0 = \sum_{j=1}^{\infty}\big[R_i\phi(t,T_i) - \alpha_i(t)\big]$$

converges uniformly for t in compact intervals a.s., cf. Basse-O'Connor and Rosiński [7, Corollary 3.2]. Moreover, β is càdlàg, see [7, Lemma 3.5], and by Lebesgue's dominated convergence theorem it follows that α_j, for $j \in \mathbb{N}$, are càdlàg as well. Therefore, with probability one,

$$\Delta X_t = \Delta\beta(t) + \sum_{j=1}^{\infty} \big[R_j \Delta\phi(t, T_j) - \Delta\alpha_j(t)\big] \quad \text{for all } t > 0.$$

Hence, for every $i \in \mathbb{N}$ almost surely

$$\Delta X_{T_i^1} = \Delta\beta(T_i^1) + \sum_{j=1}^{\infty} \big[R_j \Delta\phi(T_i^1, T_j) - \Delta\alpha_j(T_i^1)\big] = \sum_{j=1}^{\infty} R_j \Delta\phi(T_i^1, T_j) \tag{3.7}$$

Indeed, by assumption (2.5) the distribution of T_i^1 is continuous. Since β may have only a countable number of discontinuities, with probability one T_i^1 is a continuity point of β. Hence $\Delta\beta(T_i^1) = 0$ a.s. Since $(\Gamma_j)_{j\in\mathbb{N}}$ are independent of T_i^1, the argument used for β also yields $\Delta\alpha_j(T_i^1) = 0$ a.s. This proves (3.7).

Furthermore, for $i \neq j$ we get

$$\begin{aligned}\mathbb{P}(\Delta\phi(T_i^1, T_j) \neq 0) &= \int_U \mathbb{P}(\Delta\phi(T_i^1, T_j) \neq 0 \,|\, T_j = u)\, \tilde{\kappa}(du) \\ &= \int_U \mathbb{P}(\Delta\phi(T_i^1, u) \neq 0)\, \tilde{\kappa}(du) = 0\end{aligned}$$

again because $\phi(\cdot, u)$ may have only countably many jumps and the distribution of T_i^1 is continuous. If $j = i$ then

$$\Delta\phi(T_i^1, T_i) = \lim_{h\downarrow 0,\, h>0} \big[\phi(T_i^1, T_i^1, T_i^2) - \phi(T_i^1 - h, T_i^1, T_i^2)\big] = \phi(T_i^1, T_i)$$

as $\phi(t, s, v) = 0$ whenever $t < s$. This simplifies (3.7) to (3.6).

Step 3: Next we will show that $\mathbf{M}$, defined in (3.3), is a well-defined càdlàg process satisfying

$$\Delta M_{T_i^1} = \Delta X_{T_i^1} \quad \text{a.s. for all } i \in \mathbb{N}. \tag{3.8}$$

Since any semimartingale has finite quadratic variation, we get with probability one

$$\infty > \sum_{0<s\leq t} \big(\Delta X_s\big)^2 \geq \sum_{0<T_i^1\leq t} \big(\Delta X_{T_i^1}\big)^2 = \sum_{i=1}^{\infty} \big[R_i \phi(T_i^1, T_i)\big]^2 \mathbf{1}_{\{0<T_i^1\leq t\}},$$

where the last equation employs (3.6). Put for $t, r \geq 0$ and $(\epsilon, s, v) \in \{-1, 1\} \times \mathbb{R} \times V$

$$H(t; r, (\epsilon, s, v)) = R(\epsilon r h(s, v), (s, v)) \phi(s, s, v) \mathbf{1}_{\{0 < s \leq t\}}.$$

The above bound shows that for each $t \geq 0$

$$\sum_{i=1}^{\infty} |H(t; \Gamma_i, (\epsilon_i, T_i^1, T_i^2))|^2 < \infty \quad \text{a.s.}$$

That implies, by Rosiński [26, Theorem 4.1], that the following limit is finite

$$\lim_{n \to \infty} \int_0^n \mathbb{E}[\![H(t; r, (\epsilon_1, T_1^1, T_1^2))^2]\!] \, dr = \int_0^\infty \mathbb{E}[\![H(t; r, (\epsilon_1, T_1^1, T_1^2))^2]\!] \, dr.$$

Evaluating this limit we get

$$\begin{aligned}
\infty &> \int_0^\infty \mathbb{E}[\![R(\epsilon_1 r h(T_1), T_1) \phi(T_i^1, T_i) \mathbf{1}_{\{0 < T_i^1 \leq t\}}]\!]^2 \, dr \\
&= \int_0^\infty \int_{\mathbb{R} \times V} \mathbb{E}[\![R(\epsilon_1 r h(s, v), (s, v)) \phi(s, s, v) \mathbf{1}_{\{0 < s \leq t\}}]\!]^2 \, \tilde{\kappa}(ds, dv) dr \\
&= 2 \int_0^\infty \int_{\mathbb{R} \times V} \mathbb{E}[\![R(\epsilon_1 u, (s, v)) \phi(s, s, v) \mathbf{1}_{\{0 < s \leq t\}}]\!]^2 \, \kappa(ds, dv) du \\
&= \int_{\mathbb{R} \times V} \int_{\mathbb{R}} [\![x \phi(s, s, v) \mathbf{1}_{\{0 < s \leq t\}}]\!]^2 \, \rho_{(s,v)}(dx) \kappa(ds, dv) \\
&= \int_{(0,t] \times V} \int_{\mathbb{R}} \min\{|x \phi(s, s, v)|^2, 1\} \, \rho_{(s,v)}(dx) \kappa(ds, dv).
\end{aligned}$$

Finiteness of this integral in conjunction with (3.1) yield the existence of the stochastic integral

$$M_t = \int_{(0,t] \times V} \phi(s, s, v) \, \Lambda(ds, dv).$$

The fact that $\mathbf{M}$ has independent increments is obvious since Λ is independently scattered, and its continuity in probability is a consequence of (2.5). We may choose a càdlàg modification of $\mathbf{M}$, cf. [27, Theorem 11.5]. Due to the independent increments, $\mathbf{M}$ is a semimartingale if and only if its shift

component $(\zeta_t)_{t\geq 0}$ is of finite variation, cf. [17, Ch. II, Corollary 5.11], which follows from (3.1) and the fact that

$$\zeta_t = \int_{(0,t]\times V} B\big(\phi(s,s,v),(s,v)\big)\,\kappa(ds,dv), \qquad t\geq 0,$$

see [25, Theorem 2.7]. For $t\geq 0$ we can write M_t as a series using the series representation (3.5) of Λ. It follows that

$$M_t = \zeta_t + \sum_{i=1}^{\infty}\big[R_i\phi(T_i^1,T_i)\mathbf{1}_{\{0<T_i^1\leq t\}} - \gamma_j(t)\big]$$

where

$$\gamma_j(t) = \int_{\Gamma_{j-1}}^{\Gamma_j} \mathbb{E}[\![R(\epsilon_1 r h(T_1),T_1)\phi(T_1^1,T_1)\mathbf{1}_{\{0<T_j^1\leq t\}})]\!]\,dr.$$

The assumption (2.5) yields that ζ and γ_j are continuous. Moreover, by arguments as above we have $\Delta M_{T_i^1} = R_i\phi(T_i^1,T_i)$ a.s. and hence by (3.6) we obtain (3.8).

Step 4: In the following we will show the existence of a sequence $(\tau_k)_{k\in\mathbb{N}}$ of totally inaccessible stopping times such that all local martingales $\mathbf{Z}=(Z_t)_{t\geq 0}$ with respect to $\mathbb{F}^\Lambda$ are purely discontinuous and with probability one

$$\{t\geq 0:\Delta Z_t\neq 0\}\subseteq\{\tau_k:k\in\mathbb{N}\}\subseteq\{T_k^1:k\in\mathbb{N}\}. \qquad (3.9)$$

To show the above choose a sequence $(B_k)_{k\geq 1}$ of disjoint sets which generates $\mathcal{V}_0$ and for all $k\in\mathbb{N}$ let $\mathbf{U}^k=(U_t^k)_{t\geq 0}$ be given by

$$U_t^k = \Lambda((0,t]\times B_k).$$

Assumption (2.5) implies that $\mathbf{U}^k$ is a continuous in probability process with independent increments and has therefore a càdlàg modification. Hence $\mathbf{U} = \{(U_t^k)_{k\in\mathbb{N}}:t\in\mathbb{R}_+\}$ is a continuous in probability càdlàg $\mathbb{R}^{\mathbb{N}}$-valued process with no Gaussian component. Let $E=\mathbb{R}^{\mathbb{N}}\setminus\{0\}$. Then E is a Blackwell space and μ defined by

$$\mu(A) = \sharp\big\{t\in\mathbb{R}_+:(t,\Delta U_t)\in A\big\}, \qquad A\in\mathscr{B}(\mathbb{R}_+\times E)$$

is a Poisson random measure on $\mathbb{R}_+\times E$. Let ν be the intensity measure of μ. By assumption (2.5) we have that $\nu(\{t\}\times E)=0$ for all $t\geq 0$, moreover, $\mathbb{F}^\Lambda$ is the least filtration for which μ is an optional random measure. Thus

according to [17], Ch. III, Theorem 1.14(b) and the remark after Ch. III, 4.35, μ has the martingale representation property, that is for all real-valued local martingales $\mathbf{Z} = (Z_t)_{t\geq 0}$ with respect to $\mathbb{F}^\Lambda$ there exists a predictable function ϕ from $\Omega \times \mathbb{R}_+ \times E$ into $\mathbb{R}$ such that

$$Z_t = \phi * (\mu - \nu)_t, \quad t \geq 0 \tag{3.10}$$

(in (3.10) the symbol $*$ denotes integration with respect to $\mu - \nu$ as in [17]). Thus by definition, see [17, Ch. II, Definition 1.27(b)], $\mathbf{Z}$ is a purely discontinuous local martingale and $(\Delta Z_t)_{t\geq 0} = (\phi(t, \Delta U_t)\mathbf{1}_{\{\Delta U_t \neq 0\}})_{t\geq 0}$, which shows that with probability one

$$\{t \geq 0 : \Delta Z_t \neq 0\} \subseteq \{t \geq 0 : \Delta U_t \neq 0\}.$$

All real-valued continuous in probability càdlàg processes $\mathbf{Y} = (Y_t)_{t\geq 0}$ with independent increments is quasi-left continuous cf. [17, Ch. II, Corollary 5.12], and hence there exists a sequence of totally inaccessible stopping times that exhausts the jumps of $\mathbf{Y}$, cf. [17, Ch. I, Proposition 2.2]. Thus by a diagonal argument we may exhausts the jumps of $\mathbf{U}$ by a sequence of totally inaccessible stopping times $(\tau_k)_{k\in\mathbb{N}}$, that is

$$\{\tau_k : k \in \mathbb{N}\} = \{t \geq 0 : \Delta U_t \neq 0\}.$$

Arguing as in Step 2 with $\phi(t, s, v) = \mathbf{1}_{(0,t]}(s)\mathbf{1}_{B_k}(v)$ shows that with probability one

$$\Delta U_t^k = \Delta\zeta(t) + \sum_{j=1}^{\infty} \left[R_j \mathbf{1}_{\{t=T_j^1\}}\mathbf{1}_{\{T_j^2\in B_k\}} - \Delta\gamma_j(t)\right] \quad \text{for all } t > 0$$

where

$$\begin{aligned} \xi(t) &= \int_{\mathbb{R}\times V} \mathbf{1}_{\{0\leq s\leq t\}}\mathbf{1}_{\{v\in B_k\}} b(s, v)\, \kappa(ds, dv), \\ \gamma_j(t) &= \int_{\Gamma_{j-1}}^{\Gamma_j} \mathbb{E}[\![R(\epsilon_1 r h(T_1), T_1)\mathbf{1}_{\{T_j^1\leq t\}}\mathbf{1}_{\{T_j^2\in B_k\}})]\!]\, dr. \end{aligned}$$

By assumption (2.5), ξ and γ_j are continuous and hence with probability one

$$\Delta U_t^k = \sum_{j=1}^{\infty} R_j \mathbf{1}_{\{t=T_j^1\}}\mathbf{1}_{\{T_j^2\in B_k\}} \qquad \text{for all } t > 0.$$

which shows that

$$\{\tau_k : k \in \mathbb{N}\} \subseteq \{T_k^1 : k \in \mathbb{N}\} \qquad \text{a.s.}$$

and completes the proof of Step 4.

Step 5: Fix $r \in \mathbb{N}$ and let $\mathbf{X}' = (X_t')_{t\geq 0}$ be given by

$$X_t' = X_t - \sum_{s\in(0,t]} \Delta X_s \mathbf{1}_{\{|\Delta X_s|>r\}}.$$

We will show that $\mathbf{X}'$ is a special semimartingale with martingale component $\mathbf{M}' = (M_t')_{t\geq 0}$ given by

$$M_t' = \tilde{M}_t - \mathbb{E}\tilde{M}_t \quad \text{where} \quad \tilde{M}_t = M_t - \sum_{s\in(0,t]} \Delta M_s \mathbf{1}_{\{|M_s|>r\}}. \tag{3.11}$$

Recall that $\mathbf{M}$ is given by (3.3). To show above we note that $\mathbf{X}'$ is a special semimartingale since its jumps are bounded by r in absolute value; denote by $\mathbf{W}$ and $\mathbf{N}$ the finite variation and martingale compnents, respectively, in the canonical decomposition $\mathbf{X}' = X_0 + \mathbf{W} + \mathbf{N}$ of $\mathbf{X}'$. That is, we want to show that $\mathbf{N} = \mathbf{M}'$. Process $\mathbf{M}'$, given by (3.11), is obviously a martingale and by (3.8) we have for all $i \in \mathbb{N}$

$$\Delta M'_{T_i^1} = \Delta M_{T_i^1} \mathbf{1}_{\{|\Delta M_{T_i^1}|\leq r\}} = \Delta X_{T_i^1} \mathbf{1}_{\{|\Delta X_{T_i^1}|\leq r\}} = \Delta X'_{T_i^1} \qquad \text{a.s.} \tag{3.12}$$

Since $\mathbf{W}$ is predictable and τ_k is a totally inaccessible stopping time we have that $\Delta W_{\tau_k} = 0$ a.s. cf. [17, Ch. I, Proposition 2.24] and hence

$$\Delta N_{\tau_k} = \Delta X'_{\tau_k} - \Delta W_{\tau_k} = \Delta X'_{\tau_k} = \Delta M'_{\tau_k} \qquad \text{a.s.} \tag{3.13}$$

the last equality follows by (3.12) and the second inclusion in (3.9). Since $\mathbf{N}$ and $\mathbf{M}'$ are local martingales they are in fact purely discontinuous local martingale, cf. Step 3, and with probability one

$$\{t \geq 0 : \Delta N_t \neq 0\} \subseteq \{\tau_k : k \in \mathbb{N}\}, \quad \{t \geq 0 : \Delta M_t' \neq 0\} \subseteq \{\tau_k : k \in \mathbb{N}\}.$$

According to (3.13) this shows that $(\Delta N_t)_{t\geq 0} = (\Delta M_t')_{t\geq 0}$, and we conclude that $\mathbf{N} = \mathbf{M}'$ since $\mathbf{N}$ and $\mathbf{M}'$ are purely discontinuous local martingales, see [17, Ch. I, Corollary 4.19]. This completes Step 4.

Step 6: We will show that $\mathbf{A}$, given by (3.4), is a predictable càdlàg process of finite variation. By linearity, $\mathbf{A}$ is a well-defined càdlàg process. According to Step 5 the process $\mathbf{W} := \mathbf{X}' - X_0 - \mathbf{M}'$ is predictable and has càdlàg paths of finite variation. Thus with $\mathbf{V} = (V_t)_{t\geq 0}$ given by

$$V_t = \sum_{s\in(0,t]} \Delta X_s \mathbf{1}_{\{|X_s|>r\}} - \sum_{s\in(0,t]} \Delta M_s \mathbf{1}_{\{|M_s|>r\}}$$

we have by the definitions of $\mathbf{W}$ and $\mathbf{V}$ that

$$A_t = X_t - X_0 - M_t = W_t + V_t - \mathbb{E}\tilde{M}_t. \tag{3.14}$$

This shows that $\mathbf{A}$ has càdlàg sample paths of finite variation. Next we will show that $\mathbf{A}$ is predictable. Since the processes $\mathbf{W}$, $\mathbf{V}$ and $\tilde{\mathbf{M}}$ depend on the truncation level r they will be denoted $\mathbf{W}^r$, $\mathbf{V}^r$ and $\tilde{\mathbf{M}}^r$ in the following. As $r \to \infty$, $V_t^r(\omega) \to 0$ point wise in (ω, t), which by (3.14) shows that $W_t^r(\omega) - \mathbb{E}\tilde{M}_t^r \to A_t(\omega)$ point wise in (ω, t) as $r \to \infty$. For all $r \in \mathbb{N}$, $(W_t^r - \mathbb{E}\tilde{M}_t^r)_{t\geq 0}$ is a predictable process, which implies that $\mathbf{A}$ is a point wise limit of predictable processes and hence predictable. This completes the proof of Step 6.

Case 2. Λ is Gaussian: Assume that Λ is a symmetric Gaussian random measure. By Basse [2, Theorem 4.6] used on $C_t = (-\infty, t] \times V$, $\mathbf{X}$ is a special semimartingale in $\mathbb{F}^\Lambda$ with martingale component $\mathbf{M} = (M_t)_{t\geq 0}$ given by

$$M_t = \int_{(0,t]\times V} \phi(s, s, v)\, \Lambda(ds, dv), \qquad t \geq 0,$$

see [2, Equation (4.11)], which completes the proof in the Gaussian case.

Case 3. Λ is general: Let us observe that it is enough to show the theorem in the above two cases. We may decompose Λ as $\Lambda = \Lambda_G + \Lambda_P$, where Λ_G, Λ_P are independent, independently scattered random measures. Λ_G is a symmetric Gaussian random measure characterized by (2.2) with $b \equiv 0$ and $\kappa \equiv 0$ while Λ_P is given by (2.2) with $\sigma^2 \equiv 0$. Observe that

$$\mathbb{F}^\Lambda = \mathbb{F}^{\Lambda_G} \vee \mathbb{F}^{\Lambda_P},$$

which can be deduced from the Lévy-Itô decomposition used processes $\mathbf{Y} = (Y_t)_{t\geq 0}$ of the form $Y_t = \Lambda((0,t] \times B)$ where $B \in \mathcal{V}_0$. We have $\mathbf{X} = \mathbf{X}^G + \mathbf{X}^P$, where $\mathbf{X}^G$ and $\mathbf{X}^P$ are defined by (2.1) with Λ_G and Λ_P in the place of

Λ, respectively. Since $(\Lambda, \mathbf{X})$ and $(\Lambda_P - \Lambda_G, \mathbf{X}^P - \mathbf{X}^G)$ have the same distributions, the process $\mathbf{X}^P - \mathbf{X}^G$ is a semimartingale with respect to $\mathbb{F}^{\Lambda_P - \Lambda_G} = \mathbb{F}^{\Lambda_P} \vee \mathbb{F}^{-\Lambda_G} = \mathbb{F}^{\Lambda}$. Consequently, processes $\mathbf{X}^G$ and $\mathbf{X}^P$ are semimartingales with respect to $\mathbb{F}^{\Lambda}$, and so, they are semimartingales relative to $\mathbb{F}^{\Lambda_G}$ and $\mathbb{F}^{\Lambda_P}$, respectively, and the general result follows from the above two cases.

The uniqueness part: Assume that $\mathbf{X}$ has a representation of the form $\mathbf{X} = X_0 + \mathbf{M}' + \mathbf{A}'$ where $\mathbf{M}'$ is a continuous in probability semimartingale with independent increments and $\mathbf{A}'$ is a càdlàg predictable process of finite variation. With $\mathbf{M}$ and $\mathbf{A}$ given by (3.3) and (3.4) we need to show that process $\mathbf{Y}$, defined by

$$\mathbf{Y} = \mathbf{M} - \mathbf{M}' = \mathbf{A}' - \mathbf{A}, \tag{3.15}$$

is a continuous deterministic function of finite variation. The first equality in (3.15) shows that $\mathbf{Y}$ is quasi-left continuous, cf. [17, Ch. II, Corollary 4.18], and the second equality shows that $\mathbf{Y}$ is predictable, which together implies that $\mathbf{Y}$ is continuous, use [17, Ch. I, Proposition 2.24+Definition 2.25]. By the Lévy-Itô decompositions of $\mathbf{M}$ and $\mathbf{M}'$ it follows that $\mathbf{Y} = \mathbf{U} + f$ where $\mathbf{U}$ is a continuous martingale and f is a continuous deterministic function of finite variation; that f is of finite variation follows by [17, Ch. II, Corollary 5.11]. Since $\mathbf{Y}$ is of finite variation we deduce that $\mathbf{U} = 0$, that is, $\mathbf{Y} = f$. This completes the proof of the uniqueness.

The special semimartingale part: To prove the part concerning the special semimartingale property of $\mathbf{X}$ we note that the process $\mathbf{A}$ in (3.4) is a special semimartingale since it is predictable and of finite variation. Thus $\mathbf{X}$ is a special semimartingale if and only if $\mathbf{M}$ is special semimartingale. Due to its independent increments, $\mathbf{M}$ is a special semimartingale if and only if $\mathbb{E}|M_t| < \infty$ for all $t > 0$, cf. [17, Ch. II, Proposition 2.29(a)], and in that case $M_t = (M_t - \mathbb{E}M_t) + \mathbb{E}M_t$ is the canonical decomposition of $\mathbf{M}$. This completes the proof. $\square$

Remark 3.6. We conclude this section by recalling that the proof of any of the results on Gaussian semimartingales $\mathbf{X}$ mentioned in the Introduction relies on the approximations of the finite variation component $\mathbf{A}$ by discrete time Doob–Meyer decompositions $\mathbf{A}^n = (A_t^n)_{t \geq 0}$ given by

$$A_t^n = \sum_{i=1}^{[2^n t]} \mathbb{E}[X_{i2^{-n}} - X_{(i-1)2^{-n}} | \mathcal{F}_{(i-1)2^{-n}}], \qquad t \geq 0$$

and showing that the convergence $\lim_n A_t^n = A_t$ holds in an appropriate sense, see [23]. This technique does not seem effective in the non-Gaussian situation since it relies on strong integrability properties of functionals of $\mathbf{X}$.

4 Some stationary increment semimartingales

In this section we consider infinitely divisible processes which are stationary increment mixed moving averages (SIMMA). Specifically, a process $\mathbf{X} = (X_t)_{t\geq 0}$ is called a SIMMA process if it can be written in the form

$$X_t = \int_{\mathbb{R}\times V} \big[f(t-s,v) - f_0(-s,v)\big]\,\Lambda(ds,dv), \qquad t \geq 0, \tag{4.1}$$

where the functions f and f_0 are deterministic measurable such that $f(s,v) = f_0(s,v) = 0$ whenever $s<0$, and $f(\cdot,v)$ is càdlàg for all v. Λ is an independently scattered infinitely divisible random measure that is invariant under translations over $\mathbb{R}$. If V is a one-point space (or simply, there is no v-component in (4.1)) and $f_0 = 0$, then (4.1) defines a moving average (a mixed moving average for a general V, cf. [29]). If V is a one-point space and $f_0(x) = f(x) = x_+^\alpha$ for some $\alpha \in \mathbb{R}$, then $\mathbf{X}$ is a fractional Lévy process.

The finite variation property of SIMMA processes was investigated in Basse-O'Connor and Rosiński [6] and these results, together with Theorem 3.1, are crucial in our description of SIMMA semimartingales.

The random measure Λ in (4.1) is as in (2.2) but the functions b and σ^2 do not depend on s and the measure κ is a product measure: $\kappa(ds,dv) = ds\, m(dv)$ for some σ-finite measure m on V. In this case, for $A \in \mathscr{S}$ and $\theta \in \mathbb{R}$

$$\begin{aligned}&\log \mathbb{E}e^{i\theta\Lambda(A)} \\ &\quad= \int_A \Big(i\theta b(v) - \frac{1}{2}\theta^2\sigma^2(v) + \int_{\mathbb{R}} (e^{i\theta x} - 1 - iu[\![x]\!])\,\rho_v(dx)\Big)\,ds\, m(dv).\end{aligned} \tag{4.2}$$

The function B in (2.4) is independent of s, so that with $B(x,v) = B(x,(s,v))$ we have

$$B(x,v) = xb(v) + \int_{\mathbb{R}} \big([\![xy]\!] - x[\![y]\!]\big)\,\rho_v(dy), \quad x\in\mathbb{R},\ v\in V. \tag{4.3}$$

Notice that Λ satisfies (2.5) since $\kappa(ds,dv) = ds\, m(dv)$.

The SIMMA process (4.1) is a special case of (2.1) if we take $\phi(t,s,v) = f(t-s,v) - f_0(-s,v)$. Therefore, from Theorem 3.1 we obtain:

Theorem 4.1. *Let* $\mathbf{X} = (X_t)_{t\geq 0}$ *be of the form*

$$X_t = \int_{\mathbb{R}\times V} \big[f(t-s,v) - f_0(-s,v)\big]\,\Lambda(ds,dv), \qquad t \geq 0,$$

specified by (4.1)–(4.2)*, and let* B *be given by* (4.3)*. Assume that*

$$\int_V \big|B\big(f(0,v),v\big)\big|\,m(dv) < \infty. \tag{4.4}$$

Then $\mathbf{X}$ *is a semimartingale with respect to the filtration* $\mathbb{F}^\Lambda = (\mathcal{F}_t^\Lambda)_{t\geq 0}$ *if and only if*

$$X_t = X_0 + M_t + A_t, \quad t \geq 0, \tag{4.5}$$

where $\mathbf{M} = (M_t)_{t\geq 0}$ *is a Lévy process given by*

$$M_t = \int_{(0,t]\times V} f(0,v)\,\Lambda(ds,dv), \quad t \geq 0, \tag{4.6}$$

(i.e., the integral exists), and $\mathbf{A} = (A_t)_{t\geq 0}$ *is a predictable process of finite variation given by*

$$A_t = \int_{\mathbb{R}\times V} [g(t-s,v) - g(-s,v)]\,\Lambda(ds,dv) \tag{4.7}$$

where $g(s,v) = f(s,v) - f(0,v)\mathbf{1}_{\{s\geq 0\}}$*.*

Now we will give specific and closely related necessary and sufficient conditions on f and Λ that make $\mathbf{X}$ a semimartingale.

Theorem 4.2 (Sufficiency)**.** *Let* $\mathbf{X} = (X_t)_{t\geq 0}$ *be specified by* (4.1)–(4.2)*. Suppose that* (4.4) *is satisfied and that for* m*-a.e.* $v \in V$*,* $f(\cdot,v)$ *is absolutely continuous on* $[0,\infty)$ *with a derivative* $\dot f(s,v) = \frac{\partial}{\partial s} f(s,v)$ *satisfying*

$$\int_V \int_0^\infty \big(|\dot f(s,v)|^2\sigma^2(v)\big)\,ds\,m(dv) < \infty, \tag{4.8}$$

$$\int_V \int_0^\infty \int_{\mathbb{R}} \big(|x\dot f(s,v)| \wedge |x\dot f(s,v)|^2\big)\,\rho_v(dx)\,ds\,m(dv) < \infty. \tag{4.9}$$

Then $\mathbf{X}$ *is a semimartingale with respect to* $\mathbb{F}^\Lambda$*.*

Proof. We need to verify the conditions of Theorem 4.1. We see that for m-a.e. $v \in V$, $g(\cdot, v)$ is absolutely continuous on $\mathbb{R}$ with derivative $\dot{g}(s,v) = \dot{f}(s,v)$ for $s > 0$ and $\dot{g}(s,v) = 0$ for $s < 0$. By Jensen's inequality, for each fixed $t > 0$, the function

$$(s,v) \mapsto g(t-s,v) - g(-s,v) = \int_0^t \dot{g}(u-s,v)\,du,$$

when substituted for $\dot{f}(s,v)$ in (4.8)–(4.9), satisfies these conditions. Indeed, it is straightforward to verify (4.8). To verify (4.9) we use the fact that $\psi\colon u \mapsto 2\int_0^{|u|}(v \wedge 1)\,dv$ is convex and satisfies $\psi(u) \le |ux| \wedge |ux|^2 \le 2\psi(u)$. In particular, $(s,v) \mapsto g(t-s,v) - g(-s,v)$ satisfies (b) of the Introduction, and so does the function

$$(s,v) \mapsto f(0,v)\mathbf{1}_{(0,t]}(s) = g(t-s,v) - g(-s,v) - [f(t-s,v) - f(-s,v)].$$

This fact together with assumption (4.4) guarantee that $\mathbf{M}$ of Theorem 4.1 is well-defined. Then $\mathbf{A}$ is well-defined by (4.5). The process $\mathbf{A}$ is of finite variation by [6, Theorem 3.1] because $g(\cdot, v)$ is absolutely continuous on $\mathbb{R}$ and $\dot{g}(\cdot, v) = \dot{f}(\cdot, v)$ satisfies (4.8)–(4.9). $\square$

Theorem 4.3 (Necessity)**.** *Suppose that* $\mathbf{X}$ *is a semimartingale with respect to* $\mathbb{F}^\Lambda$ *and for* m*-almost every* $v \in V$ *we have either*

$$\int_{-1}^{1} |x|\,\rho_v(dx) = \infty \quad \text{or} \quad \sigma^2(v) > 0. \tag{4.10}$$

Then for m*-a.e.* v*,* $f(\cdot, v)$ *is absolutely continuous on* $[0,\infty)$ *with a derivative* $\dot{f}(\cdot, v)$ *satisfying* (4.8) *and*

$$\int_0^\infty \int_{\mathbb{R}} \big(|x\dot{f}(s,v)| \wedge |x\dot{f}(s,v)|^2\big)(1 \wedge x^{-2})\,\rho_v(dx)\,ds < \infty. \tag{4.11}$$

If, additionally,

$$\limsup_{u\to\infty} \frac{u\int_{|x|>u}|x|\,\rho_v(dx)}{\int_{|x|\le u} x^2\,\rho_v(dx)} < \infty \qquad m\text{-a.e.} \tag{4.12}$$

then for m*-a.e.* v*,*

$$\int_0^\infty \int_{\mathbb{R}} (|x\dot{f}(s,v)|^2 \wedge |x\dot{f}(s,v)|)\,\rho_v(dx)\,ds < \infty. \tag{4.13}$$

Finally, if

$$\sup_{v\in V}\sup_{u>0}\frac{u\int_{|x|>u}|x|\,\rho_v(dx)}{\int_{|x|\le u}x^2\,\rho_v(dx)}<\infty \tag{4.14}$$

then $\dot f$ *satisfies* (4.8)–(4.9).

Proof. Assume that $\mathbf{X}$ is a semimartingale with respect to $\mathbb{F}^\Lambda$. By a symmetrization argument we may assume that Λ is a symmetric random measure. Indeed, let Λ' be an independent copy of Λ and $\mathbf{X}'$ be defined by (4.1) with Λ replaced by Λ'. Then $\mathbf{X}'$ is a semimartingale with respect to $\mathbb{F}^{\Lambda'}$. By the independence, both $\mathbf{X}$ and $\mathbf{X}'$ are semimartingales with respect to $\mathbb{F}^\Lambda\vee\mathbb{F}^{\Lambda'}$ and since $\mathbb{F}^{\Lambda-\Lambda'}\subseteq\mathbb{F}^\Lambda\vee\mathbb{F}^{\Lambda'}$, the process $\mathbf{X}-\mathbf{X}'$ is a semimartingale with respect to $\mathbb{F}^{\Lambda-\Lambda'}$. This shows that we may assume that Λ is symmetric. Then (4.4) holds since $B=0$.

By Theorem 4.1 process $\mathbf{A}$ in (4.7) is of finite variation. It follows from [6, Theorem 3.3] that for m-a.e. v, $g(\cdot,v)$ is absolutely continuous on $\mathbb{R}$ with a derivative $\dot g(\cdot,v)$ satisfying (4.8) and (4.11). Furthermore $\dot g$ satisfies (4.13) under assumption (4.12), and under assumption (4.14), $\dot g$ satisfies (4.9). Since $f(s,v)=g(s,v)+f(0,v)\mathbf{1}_{\{s\ge 0\}}$, $f(\cdot,v)$ is absolutely continuous on $[0,\infty)$ with a derivative $\dot f(\cdot,v)=\dot g(\cdot,v)$ for m-a.e. v satisfying the conditions of the theorem. □

Remark 4.4. Theorem 4.3 becomes an exact converse to Theorem 4.2 when (4.10) holds and either (4.12) holds and V is a finite set, or (4.14) holds.

Remark 4.5. Condition (4.10) is in general necessary to deduce that f has absolutely continuous sections. Indeed, let V be a one point space so that Λ is generated by increments of a Lévy process denoted again by Λ. If (4.10) is not satisfied, then taking $f=\mathbf{1}_{[0,1]}$ we get that $X_t=\Lambda_t-\Lambda_{t-1}$ is of finite variation and hence a semimartingale, but f is not continuous on $[0,\infty)$.

Next we will consider several consequences of Theorems 4.2–4.3. When there is no v-component, (4.4) is always satisfied and Λ is generated by a two-sided Lévy process. In what follows, $\mathbf{Z}=(Z_t)_{t\in\mathbb{R}}$ will denote a non-deterministic two-sided Lévy process, with characteristic triplet (b,σ^2,ρ) and $Z_0=0$. $\mathbb{F}^Z$ will denote the least filtration satisfying the usual conditions such that $\sigma(Z_u:u\in(-\infty,t])\subseteq\mathcal{F}^Z_t$ for all $t\ge 0$.

The following proposition characterizes fractional Lévy processes which are semimartingales, and completes results of [3, Corollary 5.4] and parts of [9, Theorem 1].

Proposition 4.6 (Fractional Lévy processes)**.** *Let $\gamma > 0$, $x_+ := \max\{x, 0\}$ for $x \in \mathbb{R}$, $\mathbf{Z}$ be a Lévy process as above, and $\mathbf{X}$ be a fractional Lévy process defined by*

$$X_t = \int_{-\infty}^{t} \left\{ (t-s)_+^{\gamma} - (-s)_+^{\gamma} \right\} dZ_s \tag{4.15}$$

where the stochastic integrals exist. Then $\mathbf{X}$ is a semimartingale with respect to $\mathbb{F}^Z$ if and only if $\sigma^2 = 0$, $\gamma \in (0, \frac{1}{2})$ and

$$\int_{\mathbb{R}} |x|^{\frac{1}{1-\gamma}} \, \rho(dx) < \infty. \tag{4.16}$$

Proof. First we notice that, as a consequence of $\mathbf{X}$ being well-defined, $\gamma < \frac{1}{2}$ and

$$\int_{|x|>1} |x|^{\frac{1}{1-\gamma}} \, \rho(dx) < \infty. \tag{4.17}$$

Indeed, since the stochastic integral (4.15) is well-defined, [25, Theorem 2.7] shows that

$$\int_{-\infty}^{t} \int_{\mathbb{R}} \left(1 \wedge |\{(t-s)^{\gamma} - (-s)_+^{\gamma}\} x|^2\right) \rho(dx) \, ds < \infty, \qquad t \geq 0. \tag{4.18}$$

This implies that $\gamma < \frac{1}{2}$ if $\rho(\mathbb{R}) > 0$. A similar argument shows that $\gamma < \frac{1}{2}$ if $\sigma^2 > 0$, and thus, by the non-deterministic assumption on $\mathbf{Z}$, we have shown that $\gamma < \frac{1}{2}$. Putting $t = 1$ in (4.18) and using the estimate $|(1-s)^{\gamma} - (-s)_+^{\gamma}| \geq |\gamma(1-s)^{\gamma-1}|$ for $s \in (-\infty, 0]$ we get

$$\begin{aligned}
\infty &> \int_{-\infty}^{0} \int_{\mathbb{R}} \left(1 \wedge |\gamma(1-s)^{\gamma-1} x|^2\right) \rho(dx) \, ds \\
&= \int_{\mathbb{R}} \int_{1}^{\infty} \left(1 \wedge |\gamma s^{\gamma-1} x|^2\right) ds \, \rho(dx) \\
&\geq \int_{\mathbb{R}} \int_{1 \leq s \leq |\gamma x|^{\frac{1}{1-\gamma}}} ds \, \rho(dx) \geq \int_{|\gamma x|>1} \left(|\gamma x|^{\frac{1}{1-\gamma}} - 1\right) \rho(dx).
\end{aligned}$$

This shows (4.17).

Suppose that $\mathbf{X}$ is a semimartingale. If $\sigma^2 > 0$, then according to Theorem 4.3, f is absolutely continuous on $[0, \infty)$ with a derivative $\dot{f}$ satisfying

$$\int_{0}^{\infty} |\dot{f}(t)|^2 \, dt = \int_{0}^{\infty} \gamma^2 t^{2(\gamma-1)} \, dt < \infty$$

which is a contradiction and shows that $\sigma^2 = 0$. By the non-deterministic assumption on $\mathbf{Z}$ we have $\rho(\mathbb{R}) > 0$. To complete the proof of the necessity part, it remains to show that

$$\int_{|x|\le 1} |x|^{\frac{1}{1-\gamma}}\,\rho(dx) < \infty. \tag{4.19}$$

Since $\dot f(t) = \gamma t^{\gamma-1}$ for $t > 0$, we have

$$\int_0^\infty \big\{|x\dot f(t)| \wedge |x\dot f(t)|^2\big\}\,dt = C|x|^{\frac{1}{1-\gamma}} \tag{4.20}$$

where $C = \gamma^{\frac{1}{1-\gamma}}(\gamma^{-1} + (1-2\gamma)^{-1})$. In the case $\int_{|x|\le 1}|x|\,\rho(dx) < \infty$ (4.19) holds since $1 < \frac{1}{1-\gamma}$. Thus we may assume that $\int_{|x|\le 1}|x|\,\rho(dx) = \infty$, that is, (4.10) of Theorem 4.3 is satisfied. By Theorem 4.3 (4.11) and (4.20) we have

$$\int_{|x|\le 1} |x|^{\frac{1}{1-\gamma}}\,\rho(dx) \le \int_{\mathbb{R}} |x|^{\frac{1}{1-\gamma}}(1\wedge x^{-2})\,\rho(dx) < \infty$$

which completes the proof of the necessity part.

On the other hand, suppose that $\sigma^2 = 0$, $\gamma \in (0,\frac12)$ and (4.16) is satisfied. By (4.16) and (4.20), f is absolutely continuous on $[0,\infty)$ with a derivative $\dot f$ satisfying (4.9) and hence $\mathbf{X}$ is a semimartingale with respect to $\mathbb{F}^Z$, cf. Theorem 4.2. ☐

Below we will recall the conditions from [6] under which (4.12) or (4.14) hold. Recall that a measure μ on $\mathbb{R}$ is said to be regularly varying if $x \mapsto \mu([-x,x]^c)$ is a regularly varying function; see [11].

Proposition 4.7 ([6, Proposition 3.5])**.** *Condition* (4.12) *is satisfied when one of the following two conditions holds for m-almost every $v \in V$*

(i) $\int_{|x|>1} x^2\,\rho_v(dx) < \infty$ *or*

(ii) ρ_v *is regularly varying at ∞ with index $\beta \in [-2,-1)$.*

Suppose that $\rho_v = \rho$ for all v, where ρ satisfies (4.12) *and is regularly varying with index $\beta \in (-2,-1)$ at 0. Then* (4.14) *holds.*

Theorems 4.2–4.3 and Proposition 4.7 gives the following:

Corollary 4.8. *Suppose that* $\mathbf{Z} = (Z_t)_{t\in\mathbb{R}}$ *is a two-sided Lévy process as above, with paths of infinite variation on compact intervals. Let* $\mathbf{X} = (X_t)_{t\geq 0}$ *be a process of the form*

$$X_t = \int_{-\infty}^{t} \big\{ f(t-s) - f_0(-s) \big\}\, dZ_s.$$

Suppose that the random variable Z_1 *is either square-integrable or has a regularly varying distribution at* ∞ *of index* $\beta \in [-2,-1)$. *Then* $\mathbf{X}$ *is a semimartingale with respect to* $\mathbb{F}^Z$ *if and only if* f *is absolutely continuous on* $[0,\infty)$ *with a derivative* $\dot f$ *satisfying*

$$\begin{aligned} &\int_0^\infty |\dot f(t)|^2\, dt < \infty \qquad \textit{if } \sigma^2 > 0, \\ &\int_0^\infty \int_{\mathbb{R}} \big(|x\dot f(t)| \wedge |x\dot f(t)|^2 \big)\, \rho(dx)\, dt < \infty. \end{aligned} \tag{4.21}$$

Proof Corollary 4.8. The conditions imposed on Z_1 are equivalent to that ρ statisfies (i) or (ii) of Proposition 4.7, respectively, cf. [14, Theorem 1] and [27, Theorem 25.3]. Moreover, (4.10) of Theorem 4.3 is equivalent to that $\mathbf{Z}$ has sample paths of infinite variation on compacs and hence the result follows by Theorems 4.2–4.3. ☐

Example 4.9. In following we will consider $\mathbf{X}$ and $\mathbf{Z}$ given as in Corollary 4.8 where $\mathbf{Z}$ is either a stable or a tempered stable Lévy process.

(i) Stable: Assume that $\mathbf{Z}$ is a symmetric α-stable Lévy process with index $\alpha \in (1,2)$, that is, $\rho(dx) = c|x|^{-\alpha-1}\,dx$ where $c > 0$, and $\sigma^2 = b = 0$. Then $\mathbf{X}$ is a semimartingale with respect to $\mathbb{F}^Z$ if and only if f is absolutely continuous on $[0,\infty)$ with a derivative $\dot f$ satisfying

$$\int_0^\infty |\dot f(t)|^\alpha\, dt < \infty. \tag{4.22}$$

We use Corollary 4.8 to show the above. Note that $\int_{|x|\leq 1} |x|\,\rho(dx) = \infty$ and ρ is regularly varying at ∞ of index $-\alpha \in (-2,-1)$. Moreover, the identity

$$\int_{\mathbb{R}} \big(|xy| \wedge |xy|^2 \big)\, \rho(dx) = C|y|^\alpha, \qquad y \in \mathbb{R}, \tag{4.23}$$

with $C = 2c((2-\alpha)^{-1} + (\alpha-1)^{-1})$, shows that (4.21) is equivalent to (4.22). Thus the result follows by Corollary 4.8.

(ii) Tempered stable: Suppose that $\mathbf{Z}$ is a symmetric tempered stable Lévy process with indexs $\alpha \in [1,2)$ and $\lambda > 0$, i.e., $\rho(dx) = c|x|^{-\alpha-1}e^{-\lambda|x|}\,dx$ where $c > 0$, and $\sigma^2 = b = 0$. Then $\mathbf{X}$ is a semimartingale with respect to $\mathbb{F}^Z$ if and only if f is absolutely continuous on $[0,\infty)$ with a derivative $\dot{f}$ satisfying

$$\int_0^\infty \big(|\dot{f}(t)|^\alpha \wedge |\dot{f}(t)|^2\big)\,ds < \infty. \tag{4.24}$$

Again we will use Corollary 4.8. The conditions imposed on $\mathbf{Z}$ in Corollary 4.8 are satisfied due to the fact that $\int_{|x|\leq 1}|x|\,\rho(dx) = \infty$ and $\int_{|x|>1}|x|^2\,\rho(dx) < \infty$. Moreover, using the asymptotics of the incomplete gamma functions we have that

$$\int_{\mathbb{R}} \big(|xu| \wedge |xu|^2\big)\,\rho(dx) \sim \begin{cases} C_1 u^\alpha & \text{as } u \to \infty \\ C_2 u^2 & \text{as } u \to 0 \end{cases} \tag{4.25}$$

where $C_1, C_2 > 0$ are finite constants depending only on α, c and λ, and we write $f(u) \sim g(u)$ as $u \to \infty$ (resp. $u \to 0$) when $f(u)/g(u) \to 1$ as $u \to \infty$ (resp. $u \to 0$). Eq. (4.25) shows that (4.21) is equivalent to (4.24), and hence the result follows by Corollary 4.8.

Example 4.10 (Multi-stable)**.** In this example we extend Example 4.9(i) to the so called multi-stable processes, that is, we will consider $\mathbf{X}$ given by (4.1) with

$$\rho_v(dx) = c(v)|x|^{-\alpha(v)-1}\,dx$$

where $\alpha\colon V \to (0,2)$ and $c\colon V \to (0,\infty)$ are measurable functions, and $b = \sigma^2 = 0$. For $v \in V$, ρ_v is the Lévy measure of a symmetric stable distribution with index $\alpha(v)$. Assume that there exists an $r > 1$ such that $\alpha(v) \geq r$ for all $v \in V$. Then $\mathbf{X}$ is a semimartingale with respect to $\mathbb{F}^\Lambda$ if and only if for m-a.e. v, $f(\cdot,v)$ is absolutely continuous on $[0,\infty)$ with a derivative $\dot{f}(\cdot,v)$ satisfying

$$\int_V \int_0^\infty \Big(\frac{c(v)}{2-\alpha(v)}|\dot{f}(s,v)|^{\alpha(v)}\Big)\,ds\,m(dv) < \infty. \tag{4.26}$$

To show the above we will argue similarly as in Example 4.9. By the symmetry, (4.4) is satisfied. For all $v \in V$, $\int_{|x|\leq 1}|x|\,\rho_v(dx) = \infty$, which shows that (4.10) of Theorem 4.3 is satisfied. By basic calculus we have for $v \in V$ that

$$u\int_{|x|>u}|x|\,\rho_v(dx) = K(v)\int_{|x|\leq u} x^2\,\rho_v(dx) \tag{4.27}$$

where $K(v) = (2-\alpha(v))/(\alpha(v)-1)$. Since $\alpha(v) \geq r$ we have that $K(v) \leq 2/(r-1) < \infty$ which together with (4.27) implies (4.14). From (4.23) we infer that (4.9) is equivalent to (4.26), and thus Theorems 4.2–4.3 conclude the proof.

Example 4.11 (supFLP)**.** Consider $\mathbf{X} = (X_t)_{t\geq 0}$ of the form

$$X_t = \int_{\mathbb{R}\times V} \big((t-s)_+^{\gamma(v)} - (-s)_+^{\gamma(v)}\big)\,\Lambda(ds,dv), \tag{4.28}$$

where $\gamma\colon V \to (0,\infty)$ is a measurable function. Processes of the form (4.28) may be viewed as superpositions of fractional Lévy processes with (possible) different indexes; hence the name supFLP. If m-almost everywhere $\gamma \in (0,\frac{1}{2})$, $\sigma^2 = 0$ and

$$\int_V \Big(\int_{\mathbb{R}} |x|^{\frac{1}{1-\gamma(v)}}\,\rho_v(dx)\Big)\big(\tfrac{1}{2}-\gamma(v)\big)^{-1}\,m(dv) < \infty, \tag{4.29}$$

then $\mathbf{X}$ is a semimartingale with respect to $\mathbb{F}^\Lambda$. Conversely, if $\mathbf{X}$ is a semimartingale with respect to $\mathbb{F}^\Lambda$ and $\int_{|x|\leq 1}|x|\,\rho_v(dx) = \infty$ for m-a.e. v, then m-a.e. $\gamma \in (0,\frac{1}{2})$, $\sigma^2 = 0$ and

$$\int_{\mathbb{R}} |x|^{\frac{1}{1-\gamma(v)}}\,\rho_v(dx) < \infty, \tag{4.30}$$

and if in addition ρ satisfies (4.14), then (4.29) holds.

To show the above let $f(t,v) = t_+^{\gamma(v)}$ for $t \in \mathbb{R}, v \in V$. Since $f(0,v) = 0$ for all v, (4.4) is satisfied. As in Example 4.6, we observe that the conditions

$$\int_{|x|\geq 1} |x|^{\frac{1}{1-\gamma(v)}}\,\rho_v(dx) < \infty \quad \text{and} \quad \gamma(v) < \tfrac{1}{2} \quad m\text{-a.e.} \tag{4.31}$$

follow from the fact that $\mathbf{X}$ is a well-defined. For $\gamma(v) \in (0,\frac{1}{2})$, $f(\cdot,v)$ is absolutely continuous on $[0,\infty)$. By (4.20) we deduce that

$$\frac{c|x|^{\frac{1}{1-\gamma(v)}}}{\frac{1}{2}-\gamma(v)} \leq \int_0^\infty \{|x\dot f(t,v)| \wedge |x\dot f(t,v)|^2\}\,dt \leq \frac{\tilde c|x|^{\frac{1}{1-\gamma(v)}}}{\frac{1}{2}-\gamma(v)} \tag{4.32}$$

for all $x \in \mathbb{R}$, where $c, \tilde c > 0$ are finite constants not depending v and x.

By Theorem 4.2 and (4.32), the sufficient part follows. To show the necessary part assume that $\mathbf{X}$ is a semimartingale with respect to $\mathbb{F}^\Lambda$ and that $\int_{|x|\leq 1}|x|\,\rho_v(dx)=\infty$ for m-a.e. v. By Theorem 4.3, $f(\cdot,v)$ is absolutely continuous with a derivative $\dot{f}(\cdot,v)$ satisfying (4.8) and (4.11). From (4.8) we deduce that $\sigma^2=0$ m-a.e. and from (4.11) and (4.32) we infer that

$$\int_{|x|\leq 1}|x|^{\frac{1}{1-\gamma(v)}}\,\rho_v(dx)<\infty \quad m\text{-a.e. } v. \tag{4.33}$$

By (4.31)–(4.33), condition (4.30) follows. Moreover, if ρ satisfies (4.14), then Theorem 4.3 together with (4.32) show (4.29). This completes the proof.

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